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Deformations of glued G_2-manifolds

Johannes Nordström

We study how a gluing construction, which produces compact manifolds with holonomy G_2 from matching pairs of asymptotically cylindrical G_2-manifolds, behaves under deformations. We show that the gluing construction defines a smooth map from a moduli space of gluing data to the moduli space $\mathcal{M}$ of torsion-free G_2-structures on the glued manifold, and that this is a local diffeomorphism. We use this to partially compactify $\mathcal{M}$, including it as the interior of a topological manifold with boundary. The boundary points are equivalence classes of matching pairs of torsion-free asymptotically cylindrical G_2-structures.

1. Introduction

The exceptional Lie group $G_2 \subset \mathrm{SO}(7)$ also occurs as an exceptional case in the classification of Riemannian holonomy groups due to Berger [1]. A G_2-manifold is a seven-dimensional Riemannian manifold with holonomy group contained in G_2. Its metric can be defined in terms of a closed differential three-form equivalent to a torsion-free G_2-structure. Joyce [6] constructed the first compact examples of manifolds with holonomy G_2. He also proved that the moduli space $\mathcal{M}$ of torsion-free G_2-structures on a compact G_2-manifold, i.e., the quotient of the space of torsion-free G_2-structures by the identity component of the diffeomorphism group, is a smooth manifold.

A G_2-manifold is asymptotically cylindrical if it is asymptotically isometric to a product cylinder outside a compact subset. Kovalev [9] explains a gluing construction which produces a compact G_2-manifold M from a pair $M_\pm$ of asymptotically cylindrical G_2-manifolds with matching cylindrical parts, and constructs new examples of compact manifolds with holonomy G_2 by this method. Topologically, M can be considered as a generalized connected sum of M_+ and M_-. Nordström [16] shows that there is a smooth moduli space of torsion-free G_2-structures on asymptotically cylindrical G_2-manifolds, extending the result of Joyce from the compact case. This leads to the question of how deformations of a compact G_2-manifold constructed by gluing are related to deformations of the asymptotically cylindrical halves. We find that the torsion-free G_2-structures obtainable from the gluing construction form an open subset of the moduli space $\mathcal{M}$ on the compact manifold. This subset can be regarded as a neighbourhood of a boundary component for $\mathcal{M}$.

The data required for the gluing construction is a pair (φ_+, φ_-) of asymptotically cylindrical G_2-structures on M_+ and M_- which satisfies a matching condition (cf Definition 2.1), together with a large parameter $L \in \mathbb{R}^+$. L controls the length of an approximately cylindrical neck in the result of the gluing. Given such a set of gluing data $(\varphi_+, \varphi_-, L)$, the gluing construction yields a torsion-free G_2-structure $Y(\varphi_+, \varphi_-, L)$ on the connected sum M. This is uniquely defined up to diffeomorphisms, and so represents a well-defined point in the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M. We will study the local properties of a gluing map defined on a quotient $\mathcal{G}$ of a space of gluing data by a natural symmetry group. By relating $\mathcal{G}$ to the moduli spaces of torsion-free G_2-structures on the halves $M_\pm$, which are smooth manifolds, we show that $\mathcal{G}$ is smooth too. The main Theorem 2.3 states that

$$Y : \mathcal{G} \to \mathcal{M}$$

is a local diffeomorphism. This result can be compared with Donaldson and Kronheimer's description [2, Section 7.2] of deformations of anti-self-dual connections on a connected sum of a pair of smooth four-manifolds.

We also explain how to apply these results to attach a boundary to $\mathcal{M}$, forming a topological manifold $\overline{\mathcal{M}}$ with boundary, so that the boundary points correspond to ways of "pulling apart" M into a pair of asymptotically cylindrical G_2-manifolds. The results about the gluing map can therefore be interpreted as a description of a neighbourhood of a boundary component of $\mathcal{M}$. Like the statement that $\mathcal{M}$ is a manifold, this is essentially a local result. Little is known about the global properties of $\mathcal{M}$. Its local properties are also studied for instance by Karigiannis and Leung [8] and Grigorian and Yau [4], who consider in particular the curvature of a natural pseudo-Riemannian metric on $\mathcal{M}$.

The topological quantum field theory proposed by Leung [12] considers generalized connected sums of almost G_2-manifolds, i.e., seven-manifolds with G_2-structure which is not necessarily torsion-free (so the associated metric need not have holonomy in G_2) but whose defining three-form is still required to be closed. The proposed TQFT assigns invariants to compact and asymptotically cylindrical almost G_2-manifolds by counting coassociative cycles, and these invariants are expected to behave well under connected sums. It is clear that a perturbation of a connected sum of asymptotically cylindrical almost G_2-manifolds remains such a connected sum, but our result shows that this holds also when working in the category of torsion-free G_2-manifolds, where the metric has holonomy in G_2.

This paper is organized as follows. Section 2 contains background for the gluing construction of compact G_2-manifolds and precise statements of our main results. In Section 3, we discuss the topology of the glued manifold M and prove Theorem 3.1, a Hodge theory gluing result of some potential independent interest. This is used in Section 4 to compute the derivative of the gluing map, proving the main Theorem 2.3. In Section 5, we outline how to attach boundary points to $\mathcal{M}$.

2. Setup

2.1. Preliminaries

We review the preliminary definitions that are required to set up the gluing construction and state the main results. For more detailed background on G_2-manifolds see [7, Chapter 10] or [17, Chapter 8].

Recall that G_2 can be defined as the automorphism group of the normed algebra of octonions. Equivalently, G_2 is the stabilizer in $GL(\mathbb{R}^7)$ of the three-form

$$\varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*. \tag{2.1}$$

A *G_2-structure* on a manifold M^7 can therefore be defined in terms of a differential three-form φ which is equivalent to φ_0 at each point. φ_0 is a stable form in the sense of Hitchin [5], i.e., the $GL(\mathbb{R}^7)$-orbit of φ_0 is open in $\Lambda^3(\mathbb{R}^7)^*$, so the set of G_2-structures on M is open in the space of three-forms on M with respect to the uniform norm. A G_2-structure naturally defines a Riemannian metric g_φ and an orientation on M, and thus also a Levi-Civita connection ∇_φ and a Hodge star $*_\varphi$. φ is called *torsion-free* if $\nabla_\varphi \varphi = 0$. By a result of Gray this condition is equivalent to $d\varphi = d^*_\varphi \varphi = 0$. Note that this is a non-linear condition, since d^*_φ depends on φ. We call a seven-dimensional manifold M equipped with a torsion-free G_2-structure and the induced Riemannian metric a *G_2-manifold*.

The *holonomy* group of a Riemannian manifold is the group of isometries of a tangent plane generated by parallel transport around closed curves. Parallel tensor fields on the manifold correspond to invariants of the holonomy group, so it is clear that a seven-dimensional Riemannian manifold M

has holonomy $\mathrm{Hol}(M)$ contained in G_2 if and only if the metric is induced by a torsion-free G_2-structure. For a compact G_2-manifold the holonomy is exactly G_2 if and only if the fundamental group $\pi_1 M$ is finite (see [7, Proposition 10.2.2]), otherwise a finite cover of M is a Riemannian product of lower-dimensional manifolds.

On a compact G_2-manifold M the group $\mathcal{D}$ of diffeomorphisms isotopic to the identity acts on the space $\mathcal{X}$ of torsion-free G_2-structures by pull-backs. The quotient $\mathcal{M} = \mathcal{X}/\mathcal{D}$ is the moduli space of torsion-free G_2-structures. Since torsion-free G_2-structures are closed forms there is a natural projection $\mathcal{M} \to H^3(M)$ to de Rham cohomology.

Theorem 2.1 [7, Theorem 10.4.4]. *Let M be a compact G_2-manifold. Then the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M is a smooth manifold, and the map $\mathcal{M} \to H^3(M)$ is a local diffeomorphism.*

For X^6 compact, we call a G_2-structure on $X \times \mathbb{R}$ *cylindrical* if it is translation-invariant and defines a product metric. The stabilizer in G_2 of a vector in $\mathbb{R}^7$ is $\mathrm{SU}(3)$. The product of a Riemannian manifold X^6 with $\mathbb{R}$ therefore has $\mathrm{Hol}(X \times \mathbb{R}) \subseteq G_2$ if and only if $\mathrm{Hol}(X) \subseteq \mathrm{SU}(3)$, so the cross-section of a *cylindrical G_2-manifold* is always a Calabi–Yau three-fold. If we let $z^1 = x^2 + \mathrm{i}x^3, z^2 = x^4 + \mathrm{i}x^5, z^3 = x^6 + \mathrm{i}x^7$ then we can write φ_0 as

$$\varphi_0 = \Omega_0 + dx^1 \wedge \omega_0, \tag{2.2}$$

where

$$\Omega_0 = \mathrm{re}(dz^1 \wedge dz^2 \wedge dz^3), \tag{2.3}$$

$$\omega_0 = \tfrac{\mathrm{i}}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3). \tag{2.4}$$

A cylindrical G_2-structure φ on $X \times \mathbb{R}$ is, therefore, of the form

$$\varphi = \Omega + dt \wedge \omega,$$

where (Ω, ω) is a pair of forms on X point-wise equivalent to (Ω_0, ω_0). If φ is torsion-free then (Ω, ω) can be considered to define a Calabi-Yau structure (or torsion-free $\mathrm{SU}(3)$-structure) on X. This means that X has an integrable complex structure, ω is the Kähler form of a Ricci-flat Kähler metric, and Ω is the real part of a non-vanishing holomorphic $(3,0)$-form.

A non-compact manifold M is said to have *cylindrical ends* if M is written as union of two pieces M_0 and M_∞ with common boundary X, where M_0 is compact, and M_∞ is identified with $X \times \mathbb{R}^+$ by a diffeomorphism (identifying ∂M_∞ with $X \times \{0\}$). X is called the *cross-section* of M. Let t be a smooth real function on M which is the $\mathbb{R}^+$-coordinate on M_∞, and negative on the interior of M_0. A tensor field s on M is said to be *exponentially asymptotic* with rate $\delta > 0$ to a translation-invariant tensor field s_∞ on M if $\mathrm{e}^{\delta t}\|\nabla^k(s - s_\infty)\|$ is bounded on M_∞ for all $k \geq 0$, with respect to a norm defined by an arbitrary Riemannian metric on X.

A metric on M is called exponentially asymptotically cylindrical (EAC) if it is exponentially asymptotic to a product metric on $X \times \mathbb{R}$, and a G_2-structure is said to be EAC if it is exponentially asymptotic to a cylindrical G_2-structure on $X \times \mathbb{R}$. The asymptotic limit of a torsion-free EAC G_2-structure then defines a Calabi–Yau structure on the cross-section X. A diffeomorphism ϕ of M is called EAC if it is exponentially close to a product diffeomorphism $(x, t) \mapsto (\Xi(x), t + h)$ of $X \times \mathbb{R}$ in a similar sense.

The moduli space of torsion-free EAC G_2-structures on an EAC G_2-manifold M is the quotient of the space of torsion-free EAC G_2-structures (with any exponential rate) by the group of EAC diffeomorphisms of M. We will review some properties of the EAC moduli space in Subsection 4.2, but note for now that Theorem 2.1 from the compact case can be generalized.

Theorem 2.2 [16, Theorem 3.2]. *Let M be an EAC G_2-manifold. Then the moduli space of torsion-free EAC G_2-structures on M is a smooth manifold.*

2.2. Gluing construction

Let $M_\pm$ be a pair of oriented dimension seven manifolds, each with a single cylindrical end, and the same cross-section X. We assume that X is oriented so that its orientation agrees with that defined by M_+ on its boundary, and is the reverse of that defined by M_- on its boundary. This ensures that the connected sum of M_+ and M_- obtained by identifying their boundaries at infinity is oriented. Let $t_\pm$ be cylindrical coordinates on $M_\pm$, respectively.

Definition 2.1. Let $\varphi_\pm$ be torsion-free EAC G_2-structures on $M_\pm$. The pair (φ_+, φ_-) is said to *match* if their asymptotic models are $\Omega \pm dt_\pm \wedge \omega$, respectively, for some Calabi–Yau structure (Ω, ω) on X compatible with the chosen orientation. Let $\mathcal{X}_y$ be the space of such pairs.

Given $L \in \mathbb{R}^+$ let $M_\pm(L) = \{y \in M_\pm : t_\pm \leq L\}$. Identify the boundaries of $M_\pm(L)$ to form a compact smooth manifold $M(L)$, and let $j^* : X \hookrightarrow M(L)$ be the inclusion of the common boundary. $M(L)$ is independent of L up to diffeomorphism, so we will often refer to it simply as M.

For notational convenience we suppose that the cylindrical end of $M_\pm$ is given by $t_\pm > -2$. Let $\rho_\pm$ be a smooth cut-off function on $M_\pm$ which is 0 for $t_\pm < L-2$ and 1 for $t_\pm > L-1$. Let α be a closed exponentially asymptotically translation-invariant m-form on $M_\pm$. Then it can be written as $\alpha_\infty + \beta_{t_\pm} + dt_\pm \wedge \gamma_{t_\pm}$ on the cylinder, with α_∞ translation invariant, and $\beta_{t_\pm} \in \Omega^m(X)$, $\gamma_{t_\pm} \in \Omega^{m-1}(X)$ both exponentially decaying in $t_\pm$. Define an $(m-1)$-form on the cylinder by

$$\eta_\pm(\alpha) = \rho_\pm \int_{t_\pm}^{\infty} \gamma_s ds. \tag{2.5}$$

Then $\alpha + d\eta_\pm(\alpha)$ is translation-invariant on $t > L-1$.

For $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ let $\tilde{\varphi}_\pm = \varphi_\pm + d\eta_\pm(\varphi_\pm)$. Then we can define a G_2-structure $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ on $M(L)$ by $\tilde{\varphi}|_{M_\pm(L)} = \tilde{\varphi}_\pm|_{M_\pm(L)}$. Note that the choice of cut-off function in the definition of $\eta_\pm$ does not affect the cohomology class of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$.

Proposition 2.1. *There is an upper semi-continuous map $L_0 : \mathcal{X}_y \to \mathbb{R}^+$ such that for any $L > L_0$ there is a unique diffeomorphism class of torsion-free G_2-structures on $M(L)$ in a small neighbourhood of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ in its cohomology class.*

Sketch proof. The idea is that for large L the torsion of $\tilde{\varphi}(\varphi_+, \varphi_-, L)$ is very small, and the structure can be perturbed to a torsion-free one using a contraction-mapping argument. See Kovalev [9, Section 5] for details. The argument is inspired by a construction of Floer [3] (see also [11]). □

The resulting G_2-metric on $M(L)$ has an almost cylindrical "neck" of length roughly $2L$, and $\operatorname{diam} M(L) \sim 2L$ as $L \to \infty$.

Kovalev [9] constructs examples of matching pairs of EAC G_2-manifolds to which the gluing construction can be applied. An EAC version of the Calabi conjecture produces EAC manifolds with holonomy SU(3). These can be multiplied by circles S^1 to form (reducible) G_2-manifolds, which form compact irreducible G_2-manifolds (manifolds with holonomy exactly G_2) when glued together. These have different topological type from the examples constructed earlier by Joyce [6].

A future paper [10] will explain how some of the examples of compact G_2-manifolds constructed by Joyce can also be produced by gluing a pair of EAC G_2-manifolds. In some of these examples the EAC components are irreducible EAC G_2-manifolds.

2.3. Statement of results

Let M be the gluing of two EAC G_2-manifolds $M_\pm$ as above. Let $\mathcal{M}$ be the moduli space of torsion-free G_2-structures on M, and $\mathcal{M}_\pm$ the moduli spaces of torsion-free EAC G_2-structures on $M_\pm$. These are all smooth manifolds by Theorems 2.1 and 2.2.

When considering how the gluing construction behaves under deformations it is natural to look at the space of matching pairs of diffeomorphism classes of torsion-free G_2-structures on M_+ and M_-, i.e., the subset $\mathcal{M}_y \subseteq \mathcal{M}_+ \times \mathcal{M}_-$ consisting of pairs which have matching images in the moduli space of Calabi–Yau structures on X. We will use the deformation theory for EAC G_2-manifolds from [16] to show that $\mathcal{M}_y$ is a manifold. However, given a matching pair of diffeomorphism classes of EAC G_2-structures there is some ambiguity in how to glue them, since we need to choose how to identify the cylindrical ends. This means both choosing how to identify the cross-sections (this ambiguity roughly corresponds to the quotient of the automorphism group of the cross-section by a subgroup generated by elements which extend to automorphisms of M_+ or M_-), and choosing the neck length for the glued manifold. It is therefore not possible to use $\mathcal{M}_y$ itself as the domain for any sensible, single-valued map to $\mathcal{M}$. Instead we define a gluing map on a moduli space of data for the gluing construction.

Definition 2.2. A set of *gluing data* is a triple $(\varphi_+, \varphi_-, L) \in \mathcal{X}_y \times \mathbb{R}$ such that $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ and $L > L_0(\varphi_+, \varphi_-)$. Let G_0 be the space of gluing data.

G_0 is an open subset of $\mathcal{X}_y \times \mathbb{R}$. Proposition 2.1 provides a well-defined smooth map to the moduli space of torsion-free G_2-structures on M,

$$Y : G_0 \to \mathcal{M}. \tag{2.6}$$

Two sets of gluing data define essentially the same gluing operation if they are equivalent under the following action. Let $\mathcal{D}_\pm$ be the group of EAC diffeomorphisms of $M_\pm$ isotopic to the identity.

Definition 2.3. $(\phi_+, \phi_-) \in \mathcal{D}_+ \times \mathcal{D}_-$ such that $\phi_\pm$ is asymptotic to $(x, t_\pm) \mapsto (\Xi_\pm(x), t_\pm + h_\pm)$ is a *matching pair of EAC diffeomorphisms* if $\Xi_+ = \Xi_-$. Let $\mathcal{D}_y$ be the identity component of the group of such pairs.

For $(\phi_+, \phi_-) \in \mathcal{D}_y$ let $h = \frac{1}{2}(h_+ + h_-)$, and define an action on $\mathcal{X}_y \times \mathbb{R}$ by

$$\phi^* : (\varphi_+, \varphi_-, L) \mapsto (\phi_+^* \varphi_+, \phi_-^* \varphi_-, L - h). \tag{2.7}$$

There is no reason why the open set $G_0 \subseteq \mathcal{X}_y \times \mathbb{R}$ should be invariant under the action of $\mathcal{D}_y$. Nevertheless, we can define

Definition 2.4. The *moduli space of gluing data* is $\mathcal{G}_0 = G_0\mathcal{D}_y/\mathcal{D}_y$.

We can project (2.7) to an action of $\mathcal{D}_y$ on the space $\mathcal{X}_y$ of matching pairs. The quotient $\mathcal{B}$ has a natural map to $\mathcal{M}_y$. By studying this map we will deduce smoothness of $\mathcal{B}$ from the fact that $\mathcal{M}_y$ is a smooth manifold. $\mathcal{G}_0$ is obviously a fibre bundle over $\mathcal{B}$ with typical fibre $\mathbb{R}^+$, so it is a smooth manifold too. Moreover, the gluing map (2.6) really is invariant under the action of $\mathcal{D}_y$, and therefore descends to a smooth map

$$Y : \mathcal{G}_0 \to \mathcal{M}. \tag{2.8}$$

Proposition 4.3 computes the derivative of the gluing map (2.8). For each matching pair (φ_+, φ_-) the derivative is invertible at $(\varphi_+, \varphi_-, L)\mathcal{D}_y \in \mathcal{G}_0$ for all large L. Therefore, Y is a local diffeomorphism on some open subset $\mathcal{G} \subseteq \mathcal{G}_0$ whose gluing parameters are sufficiently large. This gives our main result.

Theorem 2.3. *Let M be a compact G_2-manifold constructed by gluing a matching pair $M_\pm$ of EAC G_2-manifolds. Then the gluing space $\mathcal{G}$ is a smooth manifold, and the gluing map $Y : \mathcal{G} \to \mathcal{M}$ is a local diffeomorphism.*

In the proof we will assume that $b^1(M) = 0$ in order to simplify some technical statements; for example, the map $\mathcal{B} \to \mathcal{M}_y$ is then a covering map. This is not a very restrictive assumption, since $b^1(M) = 0$ when M has holonomy exactly G_2, which is the most interesting case. In general $\mathcal{B} \to \mathcal{M}_y$ is a submersion, and the fibres have dimension $b^1(M)$.

The most important tool in the proof is to use the local diffeomorphism $\pi_H : \mathcal{M} \to H^3(M)$. This means that we can study the local properties of the gluing map in terms of what the gluing does to the cohomology classes. This is discussed in Section 3; in particular we prove a Hodge theory gluing result.

Theorem 2.3 is proved in Section 4. In Section 5, we outline how these arguments can also be used to show that $\mathcal{M}$ can be partially compactified by inclusion in a topological manifold $\overline{\mathcal{M}}$ with boundary, so that the paths defined by gluing a matching pair of EAC G_2-structures with increasing gluing parameter converge to a boundary point. The boundary points can therefore be considered as ways of "pulling apart" M into a pair of EAC connected-summands. Since the gluing space $\mathcal{G}$ is a fibre bundle over $\mathcal{B}$ with typical fibre $\mathbb{R}^+$, there is a natural way to form a fibre bundle $\overline{\mathcal{G}}$ over $\mathcal{B}$ with typical fibre $(0, \infty]$, and $\partial\overline{\mathcal{G}} = \mathcal{B}$. The partial compactification of $\mathcal{M}$ can then be described in the following way.

Theorem 2.4. *Let M be a compact G_2-manifold constructed by gluing a matching pair $M_\pm$ of EAC G_2-manifolds. Then the moduli space $\mathcal{M}$ of torsion-free G_2-structures on M can be included as the interior of a topological manifold $\overline{\mathcal{M}}$ with a boundary $\partial\overline{\mathcal{M}}$, so that the gluing map Y extends to a local homeomorphism*

$$Y : \overline{\mathcal{G}} \to \overline{\mathcal{M}}.$$

The restriction of Y to the boundary is a covering map $\partial\overline{\mathcal{G}} \to \partial\overline{\mathcal{M}}$.

3. Gluing and topology

3.1. Topology of the connected sum

Let M_+^n, M_-^n be oriented manifolds, each with a single cylindrical end, which have common cross-section X^{n-1}. As in Subsection 2.2, we assume that X is oriented compatible with M_+ and reverse to M_-, and we form a generalized connected sum M. We collect here some results about the topology of M that we will use.

As we remarked before, as a smooth manifold M is independent of the choice of gluing parameter L. Up to isotopy there are natural inclusion maps

$$\begin{tikzcd} & M & \\ M_+ \arrow[ur, "i_+"] & & M_- \arrow[ul, "i_-"'] \\ & X \arrow[ul, "j_+"] \arrow[ur, "j_-"'] \arrow[uu, "j"] & \end{tikzcd}$$

A large part of what we need to know about the topology is contained in the exactness of the Mayer–Vietoris sequence for $M = M_+ \cup M_-$ and the sequence for the cohomology of $M_\pm$ relative to its boundary X. Throughout,

H^* refers to de Rham cohomology.

(3.1)
$$\cdots H^{m-1}(X) \xrightarrow{\delta} H^m(M) \xrightarrow{i_+^* \oplus i_-^*} H^m(M_+) \oplus H^m(M_-) \xrightarrow{j_+^* - j_-^*} H^m(X) \cdots,$$

(3.2)
$$\cdots H^{m-1}(X) \xrightarrow{\partial_\pm} H^m_{\mathrm{cpt}}(M_\pm) \xrightarrow{e_\pm} H^m(M_\pm) \xrightarrow{j_\pm^*} H^m(X) \cdots$$

Here, $e_\pm$ is induced by the natural chain map $\Omega^*_{\mathrm{cpt}}(M_\pm) \to \Omega^*(M_\pm)$, and δ and $\partial_\pm$ denote the boundary homomorphisms. The inclusions $i_\pm : M_\pm \hookrightarrow M$ induce maps $i_{\pm *} : H^m_{\mathrm{cpt}}(M_\pm) \to H^m(M)$. Note that

$$\delta = i_{+*} \circ \partial_+ = -i_{-*} \circ \partial_-. \tag{3.3}$$

$j_\pm^* : H^m(M_\pm) \to H^m(X)$ is the Poincaré dual of $\pm\partial_\pm : H^{n-m-1}(X) \to H^{n-m}(M)$ (the sign difference comes from our assumption on the orientations of $M_\pm$ and X). The Poincaré dual of the Mayer–Vietoris sequence is the sequence for the relative cohomology of (M, X),

(3.4)
$$\cdots\ H^{m-1}(X) \xrightarrow{\partial_+ \oplus \partial_-} H^m_{\mathrm{cpt}}(M_+) \oplus H^m_{\mathrm{cpt}}(M_-) \xrightarrow{i_{+*} + i_{-*}} H^m(M) \xrightarrow{j^*} H^m(X)\ \cdots$$

Denote the image of $j_\pm^* : H^m(M_\pm) \to H^m(X)$ by $A^m_\pm$, and let A^m_d be the image of $j^* : H^m(M) \to H^m(X)$. By the exactness of the Mayer–Vietoris sequence, $A^m_d = A^m_+ \cap A^m_-$.

3.2. Gluing and cohomology

We explain how to glue a matching pair of closed forms on M_+, M_- to a well-defined cohomology class on M.

Let $\mathcal{Z}^m_y$ be the space of matching pairs of closed exponentially asymptotically translation-invariant m-forms on M_+, M_-, i.e., (ψ_+, ψ_-) such that $\psi_\pm$ is a closed exponentially asymptotically translation-invariant m-form on $M_\pm$, with asymptotic limits $B_a(\psi) \pm dt_\pm \wedge B_e(\psi)$, respectively.

If $(\psi_+, \psi_-) \in \mathcal{Z}^m_y$ and $L > 0$ let $\tilde{\psi}_\pm = \psi_\pm + d\eta_\pm(\psi_\pm)$. Choose the cut-off function for the cylinders in definition (2.5) of $\eta_\pm$ to ensure that $\tilde{\psi}_\pm$ is translation-invariant on $t_\pm > 0$. Then we can define $\tilde{\psi}(\psi_+, \psi_-, L)$ on $M(L)$ by $i_\pm^* \tilde{\psi} = \tilde{\psi}_\pm$. We define a gluing map

$$Y_H : \mathcal{Z}^m_y \times \mathbb{R}^+ \to H^m(M), \ \ (\psi_+, \psi_-, L) \mapsto [\tilde{\psi}]. \tag{3.5}$$

Y_H is independent of the choice of $\eta_\pm$ and hence well-defined. Furthermore, we find that Y_H is invariant under the action of the group $\mathcal{D}_y$ of matching diffeomorphisms from Definition 2.3.

Definition 3.1. For $(\phi_+, \phi_-) \in \mathcal{D}_y$ with asymptotic models $(x, t_\pm) \mapsto (\Xi(x), t_\pm + h_\pm)$ let $h = \frac{1}{2}(h_+ + h_-)$, and define an action on $\mathcal{Z}_y^m \times \mathbb{R}$ by

$$\phi^* : (\psi_+, \psi_-, L) \mapsto (\phi_+^* \psi_+, \phi_-^* \psi_-, L - h). \tag{3.6}$$

Proposition 3.1. *If $(\psi_+, \psi_-, L) \in \mathcal{Z}_y^m \times \mathbb{R}^+$, and $(\phi_+, \phi_-) \in \mathcal{D}_y$ with $h_\pm < L$ then*

$$Y_H(\psi_+, \psi_-, L) = Y_H(\phi_+^* \psi_+, \phi_-^* \psi_-, L - h) \in H^m(M).$$

Sketch of proof. Let $\tilde{\psi} = \tilde{\psi}(\psi_+, \psi_-, L)$ and $\tilde{\psi}' = \tilde{\psi}(\phi_+^* \psi_+, \phi_-^* \psi_-, L - h)$. ϕ_+ and ϕ_- can be approximately glued to a diffeomorphism $\tilde{\phi} : M(L-h) \to M(L)$ which pulls back $[\tilde{\psi}]$ to $[\tilde{\psi}']$. $\square$

Proposition 3.2. *If $(\psi_+, \psi_-) \in \mathcal{Z}_y^m$ with $B_e(\psi) = \tau$, $L, h \in \mathbb{R}^+$ then*

$$Y_H(\psi_+, \psi_-, L + h) = Y_H(\psi_+, \psi_-, L) + 2h\delta([\tau]), \tag{3.7}$$

where δ is the boundary homomorphism appearing in the Mayer–Vietoris sequence (3.1)*.*

Proof. It suffices to prove the result separately for the cases when $B_a(\psi) = 0$ and $B_e(\psi) = 0$.

If $B_e(\psi) = 0$ pick a diffeomorphism $f : (0, L) \to (0, L + h)$ which is id on $(0, 1)$ and $\mathrm{id} + h$ on $(L - 1, L)$. We can define a diffeomorphism $M(L) \to M(L + h)$ which is the identity on the images of the compact pieces of M_+ and M_- in $M(L)$ and $(x, t) \mapsto (x, f(t))$ on the cylindrical part. This pulls back $\tilde{\psi}(\psi_+, \psi_-, L) \mapsto \tilde{\psi}(\psi_+, \psi_-, L + h)$.

If $B_a(\psi) = 0$ let $c_\pm = \pm\tilde{\psi}_\pm - d(\rho_\pm t_\pm \tau)$, with $\rho_\pm$ a cut-off function chosen so that $c_\pm$ has support contained in $t_\pm < 1$. By definition of the Mayer–Vietoris boundary map δ, the form on $M(L)$ obtained by gluing $d(\rho_+ t_+ \tau)$ and $-d(\rho_- t_- \tau)$ is cohomologous to $\delta((t_+ + t_-)[\tau]) = 2L\delta([\tau])$ for any L. Hence for any L

$$Y_H(\psi_+, \psi_-, L) = i_{+*}([c_+]) + i_{-*}(-[c_-]) + 2L\delta([\tau]). \tag{3.8}$$

Since $i_{\pm *} : H^m_{\mathrm{cpt}}(M_\pm) \to H^m(M)$ and $\delta : H^{m-1}(X) \to H^m(M)$ are independent of L the result follows. $\square$

It is convenient to use Proposition 3.2 to extend Y_H to negative gluing parameters in a well-defined way.

Definition 3.2. Define

$$Y_H : \mathcal{Z}^m_y \times \mathbb{R} \to H^m(M)$$

as (3.5) on $\mathcal{Z}_y \times \mathbb{R}^+$, and extend for any $L > 0$ and $h \in \mathbb{R}$ by (3.7).

3.3. EAC Hodge theory

Let $M_\pm$ be an EAC manifold with cross-section X. We summarize the Hodge theory from [16, Section 5] (see also [13, Section 6.4]).

Let $\mathcal{H}^m_{\pm,0}$ be the bounded harmonic forms on $M_\pm$. This space has finite dimension, and its elements are smooth, closed, co-closed and exponentially asymptotically translation-invariant. The asymptotic limit of $\beta \in \mathcal{H}^m_{\pm,0}$ is a translation-invariant harmonic form on the cylinder $X \times \mathbb{R}$, so if $\mathcal{H}^m_X$ denotes the space of harmonic m-forms on X then the limit can be written as

$$B_\pm(\beta) = B_{\pm,a}(\beta) + dt_\pm \wedge B_{\pm,e}(\beta) \in \mathcal{H}^m_X + dt_\pm \wedge \mathcal{H}^{m-1}_X.$$

Note that

$$j^*_\pm[\beta] = [B_{\pm,a}(\beta)] \in H^m(X).$$

The image of $B_{\pm,a} : \mathcal{H}^m_{\pm,0} \to \mathcal{H}^m_X$ is therefore precisely the space of harmonic representatives $\mathcal{A}^m_\pm$ of the cohomology classes $A^m_\pm \subseteq H^m(X)$. Let $\mathcal{H}^m_{\pm,\mathrm{abs}} = \ker B_{\pm,e}$, and $\mathcal{H}^m_E$ the subset of exact forms in $\mathcal{H}^m_{\pm,0}$. Then the natural map

$$\mathcal{H}^m_{\pm,\mathrm{abs}} \to H^m(M)$$

is an isomorphism, and

$$\mathcal{H}^m_{\pm,0} = \mathcal{H}^m_{\pm,\mathrm{abs}} \oplus \mathcal{H}^m_{\pm,E}.$$

$B_{\pm,e}$ maps $\mathcal{H}^m_{\pm,E}$ isomorphically to its image $\mathcal{E}^m_\pm$. Further

$$\mathcal{H}^m_X = \mathcal{A}^m_\pm \oplus \mathcal{E}^m_\pm,$$

and this direct sum is orthogonal.

3.4. Hodge theory and gluing

Now suppose that $M_\pm$ are EAC Riemannian manifolds whose cylindrical models match. We wish to consider what the gluing of closed forms described in Subsection 3.2 does on matching pairs of harmonic forms, i.e., on the space

$$\mathcal{H}^m_y = (\mathcal{H}^m_{+,0} \times \mathcal{H}^m_{-,0}) \cap \mathcal{Z}^m_y.$$

We prove that any cohomology class on M can be obtained by gluing a matching pair of harmonic forms with a fixed gluing parameter L, except when L corresponds to an eigenvalue of a certain endomorphism that we will define below.

Theorem 3.1. *Let M_+, M_- have EAC metrics. Considering L as a parameter, the linear map*

$$Y_H : \mathcal{H}^m_y \to H^m(M), \;\; (\psi_+, \psi_-) \mapsto Y_H(\psi_+, \psi_-, L) \tag{3.9}$$

is an isomorphism except when $-2L$ is an eigenvalue of

$$\pi_E(\partial_+^{-1}C_+ + \partial_-^{-1}C_-) : E^{m-1}_d \to E^{m-1}_d. \tag{3.10}$$

We can write $H^m(X)$ as an orthogonal direct sum $A^m_\pm \oplus E^m_\pm$, where $A^m_\pm$ is the image of $j^*_\pm : H^m(M_\pm) \to H^m(X)$.

Let $\mathcal{A}^m_d = \mathcal{A}^m_+ \cap \mathcal{A}^m_-$. This is then the space of harmonic representatives for A^m_d. Similarly let $\mathcal{E}^m_d = \mathcal{E}^m_+ \cap \mathcal{E}^m_-$, and denote the corresponding subspace of $H^m(X)$ by E^m_d. Let $\pi_E : H^m(M) \to E^m_d$ denote the L^2-orthogonal projection.

Recall that $\partial_\pm$ denotes the boundary map in the long exact sequence for relative cohomology (3.2). It is convenient to define an isomorphism

$$C_\pm : E^{m-1}_\pm \to \operatorname{im} \partial_\pm \subseteq H^m_{\mathrm{cpt}}(M_\pm)$$

as follows. For $\tau \in \mathcal{E}^{m-1}_\pm$ let ψ be the unique element of $\mathcal{H}^m_{\pm,E}$ (the bounded exact harmonic forms on $M_\pm$) such that $B_{\pm,e}(\psi) = \tau$. If we take $\eta_\pm$ as defined in (2.5) and $\rho_\pm$ a cut-off function for the cylinder of $M_\pm$ then $\psi + d\eta_\pm(\psi) - d(\rho_\pm t_\pm \tau)$ has compact support, so represents a class $C_\pm([\tau]) \in H^m_{\mathrm{cpt}}(M_\pm)$. This class is mapped to 0 by $e_\pm$, so lies in the image of $\partial_\pm$. Composing $C_\pm$ with the inverse of $\partial_\pm : E^{m-1}_\pm \to \operatorname{im} \partial_\pm$ gives an endomorphism $\partial_\pm^{-1} C_\pm$ of $E^{m-1}_\pm$.

Remark 3.1. $\partial_\pm^{-1} C_\pm : E_\pm^m \to E_\pm^m$ is self-adjoint, and hence so is the endomorphism (3.10). $C_\pm$ is independent of the choice of $\rho_\pm$, but depends on both the metric and the cylindrical coordinate — replacing $t_\pm$ by $t_\pm + \lambda$ adds $\lambda\partial_\pm$ to $C_\pm$.

Proof of Theorem 3.1. Consider the map $(i_+^* \oplus i_-^*) : H^m(M) \to H^m(M_+) \oplus H^m(M_-)$ in the Mayer–Vietoris sequence. Recall that L is fixed, so that Y_H is regarded as a linear map $\mathcal{H}_y^m \to H^m(M)$. To show that it is an isomorphism it suffices to show that $\operatorname{im}((i_+^* \oplus i_-^*) \circ Y_H) = \operatorname{im}(i_+^* \oplus i_-^*)$, and that $Y_H : \ker((i_+^* \oplus i_-^*) \circ Y_H) \to \ker(i_+^* \oplus i_-^*)$ is an isomorphism.

$(i_+^* \oplus i_-^*) Y_H(\psi_+, \psi_-) = ([\psi_+], [\psi_-])$ and it follows from the exactness of the Mayer–Vietoris sequence that $\operatorname{im}((i_+^* \oplus i_-^*) \circ Y_H) = \operatorname{im}(i_+^* \oplus i_-^*)$. It also follows that $\ker((i_+^* \oplus i_-^*) \circ Y_H) = \mathcal{H}_{y,E}^m$, the pairs of exact forms in $\mathcal{H}_y^m$.

Thus the problem reduces to determining whether the restriction

$$Y_H : \mathcal{H}_{y,E}^m \to \ker(i_+^* \oplus i_-^*)$$

of (3.9) is an isomorphism. Given $\tau \in \mathcal{E}_d^{m-1}$ let (ψ_+, ψ_-) be the unique element of $\mathcal{H}_{y,E}^m$ such that $\tau = B_{+,e}(\psi_+) = -B_{-,e}(\psi_-)$. By the definition of $C_\pm$ and (3.8)

$$Y_H(\psi_+, \psi_-) = i_{+*} C_+([\tau]) + i_{-*} C_-([-\tau]) + 2L\delta([\tau]).$$

Combining with (3.3)

$$Y_H(\psi_+, \psi_-) = \delta\left(\partial_+^{-1} C_+([\tau]) + \partial_-^{-1} C_-([\tau]) + 2L[\tau]\right). \tag{3.11}$$

$\delta : H^{m-1}(X) \to H^m(M)$ is an isomorphism $E_d^{m-1} \to \ker(i_+^* \oplus i_-^*)$ and vanishes on the orthogonal complement of E_d^{m-1}. It follows that (3.11) gives an isomorphism $\mathcal{H}_{y,E}^m \to \ker(i_+^* \oplus i_-^*)$ unless $-2L$ is an eigenvalue of the endomorphism (3.10). □

4. The gluing map

We will now make use of the topological results of the previous section to study the gluing map for torsion-free G_2-structures. As in Section 2 the setup is that M_+ and M_- are EAC G_2-manifolds with a common cross-section X, and M is their connected sum. $\mathcal{M}$ denotes the moduli space of torsion-free G_2-structures on M, and G_0 the space of gluing data.

In order to prove Theorem 2.3 we need to show that the gluing map is invariant under $\mathcal{D}_y$ (the identity component of the group of matching pairs of EAC diffeomorphisms of M_+ and M_-) so that it is well-defined on $\mathcal{G}_0 = G_0\mathcal{D}_y/\mathcal{D}_y$, show that $\mathcal{G}_0$ is a smooth manifold, and compute the derivative of the gluing map.

4.1. Diffeomorphism invariance

Note that the composition $\pi_H \circ Y : G_0 \to H^3(M)$ of the gluing map (2.6) with the local diffeomorphism $\pi_H : \mathcal{M} \to H^3(M)$ is simply the restriction to G_0 of the map Y_H given by Definition 3.2. We will use this first to show that Y induces a well-defined map on the quotient $\mathcal{G}_0$. Later we will determine the local properties of $Y : \mathcal{G}_0 \to \mathcal{M}$ from those of $Y_H : (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y \to H^3(M)$.

Proposition 4.1. *The map $Y : G_0 \to \mathcal{M}$ is $\mathcal{D}_y$-invariant, so descends to a well-defined continuous function*

$$Y : \mathcal{G}_0 \to \mathcal{M}. \tag{4.1}$$

Proof. We need to show that if $\phi \in \mathcal{D}_y$ and $(\varphi_+, \varphi_-, L) \in G_0$ such that $\phi^*(\varphi_+, \varphi_-, L) \in G_0$ then

$$Y(\varphi_+, \varphi_-, L) = Y(\phi^*(\varphi_+, \varphi_-, L)).$$

The idea of the proof is to connect $(\varphi_+, \varphi_-, L)$ and $\phi^*(\varphi_+, \varphi_-, L)$ by a path in G_0. The image under Y of this path is the lift by the local diffeomorphism $\pi_H : \mathcal{M} \to H^3(M)$ of a path in $H^3(M)$, which is determined by Propositions 3.1 and 3.2.

Let $[0, 1] \to \mathcal{M}$, $s \mapsto \phi_s$ be a path in $\mathcal{D}_y$ connecting the identity to ϕ, and take k sufficiently large that $\phi_s^*(\varphi_+, \varphi_-, L + k) \in G_0$ for all s. By Proposition 3.1 the path $s \mapsto Y(\phi_s^*(\varphi_+, \varphi_-, L + k)) \in \mathcal{M}$ is a lift of a constant path in $H^3(M)$, so

$$Y(\phi^*(\varphi_+, \varphi_-, L + k)) = Y((\varphi_+, \varphi_-, L + k)).$$

By Proposition 3.2 the paths

$$\begin{gathered} s \mapsto Y(\phi^*(\varphi_+, \varphi_-, L + (1 - s)k)) \in \mathcal{M}, \\ s \mapsto Y((\varphi_+, \varphi_-, L + (1 - s)k)) \in \mathcal{M} \end{gathered}$$

are both lifts of $s \mapsto Y_H(\varphi_+, \varphi_-, L + k) - 2ks\delta([\omega])$, so in particular they have the same value at $s = 1$, which gives the result. ∎

4.2. Deformations of EAC G_2-manifolds

In order to define coordinate charts for $\mathcal{G}_0$ we first summarize the deformation theory for EAC G_2-manifolds developed in [16, Section 6]. Let $\mathcal{X}_\pm$ be the space of torsion-free EAC G_2-structures on $M_\pm$ (with any exponential rate) and $\mathcal{D}_\pm$ the group of EAC diffeomorphisms isotopic to the identity. Then the moduli space $\mathcal{M}_\pm = \mathcal{X}_\pm/\mathcal{D}_\pm$ is a smooth manifold.

Any EAC torsion-free G_2-structure $\varphi_\pm$ on $M_\pm$ is asymptotic to $\Omega \pm dt_\pm \wedge \omega$, where (Ω, ω) is a Calabi–Yau structure on X. This defines a natural boundary map $B_\pm : \mathcal{M}_\pm \to \mathcal{N}$, where $\mathcal{N}$ is the moduli space of Calabi–Yau structures on X. Since $[\Omega] = j_\pm^*[\varphi_\pm]$ and $\frac{1}{2}[\omega]^2 = j_\pm^*[*\varphi_\pm]$ it is clear that any element in the image of the boundary map satisfies

$$[\Omega] \in A_\pm^3, \quad [\omega]^2 \in A_\pm^4, \tag{4.2}$$

where $A_\pm^m = \mathrm{im}(j_\pm^* : H^m(M_\pm) \to H^m(X))$ as before. These conditions define a subset $\mathcal{N}_{\pm,A} \subseteq \mathcal{N}$. The boundary map $B_\pm$ is a submersion onto its image, which is a submanifold of $\mathcal{N}$ and an open subset of $\mathcal{N}_{\pm,A}$.

The proof of these results uses *pre-moduli spaces* as coordinate charts. There is a manifold $\mathcal{R}_\pm$ of torsion-free EAC G_2-structures near $\varphi_\pm$, such that the natural map $\mathcal{R}_\pm \to \mathcal{M}_\pm$ is a homeomorphism onto an open subset. The transition function between such maps is smooth, so they can be used as coordinate charts. Similarly, there is a manifold $\mathcal{Q}$ of Calabi–Yau structures near (Ω, ω) such that $\mathcal{Q} \to \mathcal{N}$ is a coordinate chart.

The subset $\mathcal{Q}_{\pm,A} \subseteq \mathcal{Q}$ defined by Equations (4.2) is a submanifold, and the boundary map

$$B_\pm : \mathcal{R}_\pm \to \mathcal{Q}$$

is a submersion onto $\mathcal{Q}_{\pm,A}$. Any tangent (σ, τ) at (Ω, ω) to a curve of SU(3)-structures satisfies the algebraic relations

$$\sigma \wedge *\Omega - \tau \wedge \omega^2 = 0, \tag{4.3}$$

$$\sigma \wedge \omega + \Omega \wedge \tau = 0. \tag{4.4}$$

The tangent space to $\mathcal{Q}$ consists of the harmonic tangents to the space of SU(3)-structures, i.e.,

$$T_{(\Omega,\omega)}\mathcal{Q} = \mathcal{H}_{\mathrm{SU}} = \{(\sigma, \tau) \in \mathcal{H}_X^3 \times \mathcal{H}_X^2 : (4.3) \text{ and } (4.4) \text{ hold}\}.$$

The tangent space to $\mathcal{R}_\pm$ consists of harmonic forms asymptotic to elements of $\mathcal{H}_{\mathrm{SU}}$,

$$T_{\varphi_\pm}\mathcal{R}_\pm = \mathcal{H}^3_{\pm,\mathrm{cyl}} = \{\psi \in \mathcal{H}^3_{\pm,0} : B_\pm(\psi) \in \mathcal{H}_{\mathrm{SU}}\}.$$

4.3. A coordinate chart

Next we describe coordinate charts for $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$, which contains $\mathcal{G}_0$ as an open subset. $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$ is a principal $\mathbb{R}$-bundle over $\mathcal{B} = \mathcal{X}_y/\mathcal{D}_y$, so it suffices to show that $\mathcal{B}$ is a manifold.

Let $\mathcal{M}_\pm$ be the moduli space of torsion-free EAC G_2-structures on $M_\pm$, and $\mathcal{N}$ the moduli space of Calabi–Yau structures on X. Let $\mathcal{M}_y \subseteq \mathcal{M}_+ \times \mathcal{M}_-$ be the pairs of diffeomorphism classes of EAC torsion-free G_2-structures whose boundary images in $\mathcal{N}$ match.

Proposition 4.2. *Let M be the gluing of a pair of EAC G_2-manifolds $M_\pm$. If $b^1(M) = 0$ then the natural projection*

$$\mathcal{B} \to \mathcal{M}_y, \;\; (\varphi_+, \varphi_-)\mathcal{D}_y \mapsto (\varphi_+\mathcal{D}_+, \varphi_-\mathcal{D}_-) \tag{4.5}$$

is a local homeomorphism.

First, we find charts for $\mathcal{M}_y$.

Proposition 4.3. *$\mathcal{M}_y$ is a submanifold of $\mathcal{M}_+ \times \mathcal{M}_-$.*

Each point in $\mathcal{M}_y$ can be represented by a matching pair of torsion-free G_2-structures (φ_+, φ_-), asymptotic to a Calabi–Yau structure (Ω, ω) on X. Let $\mathcal{R}_\pm$ be the pre-moduli space of torsion-free EAC G_2-structures near $\varphi_\pm$.

Definition 4.1. The pre-moduli space of matching pairs of torsion-free EAC G_2-structures near (φ_+, φ_-) is a neighbourhood $\mathcal{R}_y$ of (φ_+, φ_-) in $\mathcal{X}_y \cap (\mathcal{R}_+ \times \mathcal{R}_-)$.

To use $\mathcal{R}_y$ as a coordinate chart we first need to show that its image under the boundary map is a manifold. The intersection

$$\mathcal{Q}_{d,A} = \mathcal{Q}_{+,A} \cap \mathcal{Q}_{-,A}$$

consists of $(\Omega', \omega') \in \mathcal{Q}$ such that $[\Omega'] \in A^3_d$, $[\omega'^2] \in A^4_d$.

Lemma 4.1. *$\mathcal{Q}_{d,A} \subseteq \mathcal{Q}$ is a submanifold.*

Proof. The proof of proposition [16, Proposition 6.2], which states that each of $\mathcal{Q}_{\pm,A}$ is a manifold, can be recycled. □

Proof of Proposition 4.3. The group $\mathcal{D}_X$ of diffeomorphisms of X isotopic to the identity acts trivially on $\mathcal{Q}$, so for $(\psi_+, \psi_-) \in \mathcal{R}_+ \times \mathcal{R}_-$

$$B_+(\psi_+)\ \mathcal{D}_X\text{-equivalent to } B_-(\psi_-) \iff B_+(\psi_+) = B_-(\psi_-).$$

Hence $\mathcal{R}_y$ is homeomorphic to a neighbourhood of $(\varphi_+\mathcal{D}_+, \varphi_-\mathcal{D}_-)$ in $\mathcal{M}_y$, and it suffices to prove that $\mathcal{R}_y$ is a submanifold of $\mathcal{R}_+ \times \mathcal{R}_-$.

By Lemma 4.1 the image of $\mathcal{Q}_{d,A}$ in $\mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$ under the diagonal map is a submanifold. $\mathcal{R}_y \subseteq \mathcal{R}_+ \times \mathcal{R}_-$ is the inverse image of $\mathcal{Q}_{d,A} \subseteq \mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$ under the submersion $B_+ \times B_- : \mathcal{R}_+ \times \mathcal{R}_- \to \mathcal{Q}_{+,A} \times \mathcal{Q}_{-,A}$, so it is a submanifold. □

Proof of Proposition 4.2. Let $(\varphi_+, \varphi_-) \in \mathcal{X}_y$, and $\mathcal{R}_y$ the pre-moduli space of nearby matching pairs. Because $\mathcal{R}_y$ is a coordinate chart for $\mathcal{M}_y$, any element of $\mathcal{X}_y$ near (φ_+, φ_-) can be written as $(\phi_+^*\psi_+, \phi_-^*\psi_-)$, with $(\psi_+, \psi_-) \in \mathcal{R}_y$ and $\phi_\pm \in \mathcal{D}_\pm$ close to id.

Let $\mathrm{Aut}_0(X) \subset \mathcal{D}_X$ be the identity component of the subgroup of automorphisms of the Calabi–Yau manifold X (this is actually independent of $(\psi_+, \psi_-) \in \mathcal{R}_y$, cf [16, Proposition 4.5]). The matching condition for $(\phi_+^*\psi_+, \phi_-^*\psi_-)$ implies that $B(\phi_-)^{-1}B(\phi_+) \in \mathrm{Aut}_0(X)$, where $B(\phi_\pm)$ denotes the asymptotic limit of $\phi_\pm$.

$\mathrm{Aut}_0(X)$ is a closed subgroup of the isometry group of X, so it is compact (see [14]). Because X is Ricci-flat the Lie algebra of $\mathrm{Aut}_0(X)$ corresponds to the space $\mathcal{H}^1_X$ of harmonic 1-forms on X. Because X is Ricci-flat these are parallel, so the group is abelian. Similarly, the Lie algebras of the automorphism groups $\mathrm{Aut}_0(M_\pm)$ of the EAC G_2-manifolds $M_\pm$ (which are independent of $\varphi_\pm \in \mathcal{R}_\pm$) correspond to the bounded harmonic 1-forms $\mathcal{H}^1_{\pm,0}$. The image of $\mathcal{H}^1_{\pm,0}$ under the boundary map $B_\pm$ is the space of harmonic representatives of $A^1_\pm = \mathrm{im}(j^*_\pm : H^1(M_\pm) \to H^1(X))$. Each of these is a half-dimensional subspace of $H^1(X)$ according to [16, Proposition 5.15], while their intersection is $A^1_d = \mathrm{im}(j^* : H^1(M) \to H^1(X))$. As we assume $b^1(M) = 0$ it follows that $H^1(X) = A^1_+ \oplus A^1_-$. Hence $\mathrm{Aut}_0(X)$ is generated by the images $B(\mathrm{Aut}_0(M_\pm))$.

Therefore, $B(\phi_-)^{-1}B(\phi_+) = B(\phi'_-)^{-1}B(\phi'_+)$ for some $\phi'_\pm \in \mathrm{Aut}_0(M_\pm)$. Then $(\phi'_+\phi_+, \phi'_-\phi_-)$ is a matching pair of diffeomorphisms, so $(\phi_+^*\psi_+, \phi_-^*\psi_-)$ is $\mathcal{D}_y$-equivalent to (ψ_+, ψ_-). Hence the image of $\mathcal{R}_y$ is open in $\mathcal{B}$, so $\mathcal{R}_y$ can be used as a coordinate chart for $\mathcal{B}$ too. □

Because $\mathcal{R}_y$ is a coordinate chart for $\mathcal{B}$ when $b^1(M) = 0$, the natural maps

$$\mathcal{R}_y \times \mathbb{R} \to (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y \tag{4.6}$$

can then be used as local trivializations for $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$ as a principal $\mathbb{R}$-bundle.

Remark 4.1. It is possible to show that $\mathcal{B} \to \mathcal{M}_y$ is a covering map when $b^1(M) = 0$. In general the connected components of the fibres are isomorphic to $H^1(M)$.

4.4. The derivative of the gluing map

Since $\pi_H : \mathcal{M} \to H^3(M)$ is a local diffeomorphism the local behaviour of the gluing map $Y : \mathcal{G} \to \mathcal{M}$ is determined by that of $Y_H = \pi_H \circ Y$. Y_H is just the gluing map for cohomology from Definition 3.2, so can be defined on all of $(\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y$. We compute the derivative.

Proposition 4.4. *Given $(\varphi_+, \varphi_-) \in \mathcal{X}_y$ the derivative of*

$$Y_H : (\mathcal{X}_y \times \mathbb{R})/\mathcal{D}_y \to H^3(M)$$

at $(\varphi_+, \varphi_-, L)$ is bijective for all sufficiently large values of L.

Proof. We make the simplifying assumption that $b^1(M) = 0$. The tangent space to the pre-moduli space of matching torsion-free G_2-structures is

$$T_{(\varphi_+,\varphi_-)}\mathcal{R}_y = \mathcal{H}^m_{y,\mathrm{cyl}},$$

the space of matching harmonic forms (ψ_+, ψ_-) whose common boundary value $B(\psi)$ lies in $\mathcal{H}_{\mathrm{SU}}$. The condition that $b^1(M) = 0$ implies that the common boundary value of any $(\psi_+, \psi_-) \in \mathcal{H}^m_y$ automatically satisfies (4.3). $\mathcal{H}^m_{y,\mathrm{cyl}} \subset \mathcal{H}^m_y$, therefore, has codimension 1, and we can take $\{(\psi_+, \psi_-) \in \mathcal{H}^m_{y,E} : B(\psi) \in \mathbb{R}[\omega]\}$ as a direct complement. Let

$$Y'_H : \mathcal{R}_y \times \mathbb{R} \to H^3(M)$$

be the representation of Y_H in the coordinate chart (4.6), and consider the derivative

$$(DY'_H)_{(\varphi_+,\varphi_-,L)} : \mathcal{H}^3_{y,\mathrm{cyl}} \times \mathbb{R} \to H^3(M).$$

The restriction of DY'_H to $\mathcal{H}^3_{y,\mathrm{cyl}} \times 0$ is just Y_H (3.9), while on $0 \times \mathbb{R}$ it is $h \mapsto 2h\delta([\omega])$. By a slight modification of the proof of Theorem 3.1, we find that

$$(i_+^* \oplus i_-^*) \circ Y_H : \mathcal{H}^3_{y,\mathrm{cyl}} \to \mathrm{im}(i_+^* \oplus i_-^*)$$

is surjective with kernel $\mathcal{H}^m_{y,\mathrm{cyl},E} = \mathcal{H}^m_{y,\mathrm{cyl}} \cap \mathcal{H}^m_{y,E}$, and that if we identify $\mathcal{H}^m_{y,\mathrm{cyl},E} \times \mathbb{R} \leftrightarrow E^2_d$ by $(\psi_+, \psi_-, h) \mapsto [B_e(\psi) + \frac{h}{L}\omega]$ then $DY'_H : \mathcal{H}^m_{y,\mathrm{cyl},E} \times \mathbb{R} \to \ker(i_+^* \oplus i_-^*)$ is identified with

$$E^2_d \to \ker(i_+^* \oplus i_-^*), \tau \mapsto \delta(2L\tau + F(\tau))$$

for some endomorphism F of E^2_d (F is the composition of (3.10) with the projection to the orthogonal complement of $\mathbb{R}[\omega]$ in E^2_d). Hence DY'_H is an isomorphism except when $-2L$ is an eigenvalue of F. $\square$

We can define $G \subseteq G_0$ to be the subset of gluing data $(\varphi_+, \varphi_-, L)$ for which the gluing parameter L is sufficiently large to ensure invertibility of the derivative of the gluing map. The quotient $\mathcal{G} = G\mathcal{D}_y/\mathcal{D}_y$ is an open subset of $\mathcal{G}_0$, and $Y : \mathcal{G} \to \mathcal{M}$ is a local diffeomorphism. This completes the proof of Theorem 2.3.

5. Boundary points of the moduli space

To conclude we describe how to attach boundary points to the moduli space of torsion-free G_2-structures of a compact G_2-manifold M obtained by gluing, outlining a proof of Theorem 2.4.

Let the compact manifold M be the gluing of two EAC G_2-manifolds $M_\pm$ as before, and

$$Y : \mathcal{G} \to \mathcal{M}$$

the gluing map for torsion-free G_2-structures. The gluing space $\mathcal{G}$ is a fibre bundle over $\mathcal{B}$ with typical fibre $\mathbb{R}^+$. It can be considered as the interior of a topological manifold $\overline{\mathcal{G}}$ with boundary $\mathcal{B}$ "at infinity" by adding a limit point to each of the fibres. We aim to add a boundary to $\mathcal{M}$ so that Y extends to a local homeomorphism $Y : \overline{\mathcal{G}} \to \overline{\mathcal{M}}$.

Assume that $b^1(M) = 0$, and let $\mathcal{R}_y$ be the pre-moduli space of matching pairs of torsion-free EAC G_2-structures near $(\varphi_+, \varphi_-) \in \mathcal{X}_y$. We can interpret the proof of Proposition 4.3 as stating that $(i_+^* \oplus i_-^*) \circ Y_H : \mathcal{R}_y \times \mathbb{R} \to \mathrm{im}(i_+^* \oplus i_-^*)$ is a submersion, and that

$$Y_H : \mathcal{R}'_y \times (L_1, \infty) \to K \tag{5.1}$$

is a local diffeomorphism for L sufficiently large, where

$$\mathcal{R}'_y = \{(\varphi'_+, \varphi'_-) \in \mathcal{R}_y : i_\pm^*[\varphi'_\pm] = i_\pm^*[\varphi_\pm]\}$$

and $K = \{[\alpha] \in H^3(M) : i_\pm^*[\alpha] = i_\pm^*[\varphi_\pm]\}$ (an affine translate of $\delta(H^2(X)) \subseteq H^3(M)$). But we can make a stronger statement. Map (5.1) has the form

$$Y_H(\varphi'_+, \varphi'_-, L) = Y_H(\varphi'_+, \varphi'_-, 0) + 2L\delta([\omega']),$$

where ω' is the Kähler form of the common boundary value of $(\varphi'_+, \varphi'_-) \in \mathcal{R}'_y$. The second term maps out an open cone in $\delta(H^2(X))$. For large enough L_1 the second term dominates, and (5.1) is a diffeomorphism onto approximately an open affine cone in K. Hence

$$Y : \mathcal{R}_y \times (L_1, \infty) \to \mathcal{M} \tag{5.2}$$

is not just a local diffeomorphism, but a diffeomorphism onto its image for large L_1. Since $\mathcal{R}_y$ are coordinate charts for $\mathcal{B}$, one could try to use (5.2) as coordinate charts to make $\mathcal{M} \cup \mathcal{B}$ a manifold with boundary. The problem is that the resulting topology need not be Hausdorff; different points of $\mathcal{B}$ could a priori arise as the limit of the same path in $\mathcal{M}$. This difficulty can be resolved by proving that the property of "defining the same boundary point" is an equivalence relation on $\mathcal{B}$ and that the quotient $\hat{\mathcal{B}}$ is covered by $\mathcal{B}$. Then $\hat{\mathcal{B}}$ is a manifold, and one can use $\overline{\mathcal{M}} = \mathcal{M} \cup \hat{\mathcal{B}}$ in the statement of Theorem 2.4.

This outline can be expanded to a full proof of Theorem 2.4. The details can be found in [15, Section 6.4], but are not included here as they amount to a rather tedious inspection of the charts (5.2).

Remark 5.1. We could give the topological manifolds $\overline{\mathcal{G}}$ and $\overline{\mathcal{M}}$ smooth structures by choosing an identification of $(0, \infty]$ with a half-open interval $[0, 1)$, but it is not clear if there is a natural choice of smooth structure.

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References

[1] M. Berger, *Sur les groupes d'holonomie homogène des variétés à connexion affines et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279–330.

[2] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, Oxford 1997.

[3] A. Floer, *Self-dual conformal structures on $l\mathbb{C}P^2$*, J. Diff. Geom. **33** (1991), 551–573.

[4] S. Grigorian and S.-T. Yau, *Local geometry of the G_2 moduli space*, Commun. Math. Phys. **287** (2009), 459–488.

[5] N.J. Hitchin, *Stable forms and special metrics*, M. Fernández and J.A. Wolf, editors, Global Differential Geometry: The Mathematical Legacy of Alfred Gray, vol **288** *Contemporary Mathematics*, American Mathematical Society, Providence, 2001, 70–89.

[6] D.D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2, I.* J. Diff. Geom. **43** (1996), 291–328.

[7] D.D. Joyce, *Compact manifolds with special holonomy*, OUP *Mathematical Monographs Series*, Oxford University Press, Oxford 2000.

[8] S. Karigiannis and N.C. Leung, *Hodge theory for G_2-manifolds: Intermediate Jacobians and Abel–Jacobi maps*, Proc. London Math. Soc. **99**(2) (2009), 297–325. Preprint, arXiv:0709.2987.

[9] A.G. Kovalev, *Twisted connected sums and special Riemannian holonomy*, J. Reine Angew. Math. **565** (2003), 125–160.

[10] A.G. Kovalev and J. Nordström, *Asymptotically cylindrical 7-manifolds of holonomy G_2 with applications to compact irreducible G_2-manifolds.* Preprint, arXiv:0907.0497.

[11] A.G. Kovalev and M.A. Singer, *Gluing theorems for complete anti-self-dual spaces*, Geom. Funct. Anal. **11** (2001), 1229–1281.

[12] N.C. Leung, *Topological quantum field theory for Calabi–Yau threefolds and G_2-manifolds*, Adv. Theor. Math. Phys. **6** (2002), 575–591.

[13] R. Melrose, *The Atiyah–Patodi–Singer index theorem*, AK Peters, Wellesley, MA, 1994.

[14] S.B. Myers and N.E. Steenrod, *The group of isometries of a riemannian manifold*, *Ann. Math.* **40** (1939), 400–416.

[15] J. Nordström, *Deformations and gluing of asymptotically cylindrical manifolds with exceptional holonomy*, PhD thesis, University of Cambridge, 2008.

[16] J. Nordström, *Deformations of asymptotically cylindrical G_2-manifolds*, Math. Proc. Cambridge Philos. Soc. **145** (2008), 311–348.

[17] S.M. Salamon, *Riemannian Geometry and holonomy groups*, Pitman Reseach Notes in Mathematics, **201**, Longman, Harlow, 1989.

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